One-dimensional Schrödinger operators with δ'-interactions on a set of Lebesgue measure zero

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Abstract

We give an abstract definition of a one-dimensional Schrödinger operator with δ'-interaction on an arbitrary set Γ of Lebesgue measure zero. The number of negative eigenvalues of such an operator is at least as large as the number of those isolated points of the set Γ that have negative values of the intensity constants of the δ'-interaction. In the case where the set Γ is endowed with a Radon measure, we give constructive examples of such operators having an infinite number of negative eigenvalues.

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1 Introduction

One important problem in the theory of singular perturbations of a Schrödinger operator is to construct non–trivial self–adjoint operators that describe interactions on a set Γ of Lebesgue measure zero [3,4]. The most studied case is the one where Γ consists of isolated points. In this case the corresponding interaction is called point interaction and leads to solvable models in quantum mechanics [3,4].

For an arbitrary closed set Γ of Lebesgue measure zero, the Schrödinger operator with interaction on Γ is defined as a self-adjoint extension of the minimal operator $-\frac{d^2}{dx^2}$ defined on functions in the space $C_0^\infty(R^1 \setminus \Gamma)$ [3,4,8,24], In some cases, other definitions of the Schrödinger operator with interaction on Γ are possible. Such definitions are given in terms of certain boundary conditions [3,4], singular perturbations [4,5], quadratic forms [1,14], construction of BVS [22,23], and other methods [30]. If Γ is endowed with a Radon measure, then Schrödinger operators with interactions on Γ can be defined using analogues of the usual boundary conditions on Γ [8,24].

In this paper, we give an abstract definition of a Schrödinger operator $L_{\Gamma,\delta'}$ with δ'-interaction on an arbitrary set Γ of Lebesgue measure zero. If the set Γ contains isolated points, then functions from the domain of such an operator satisfy the usual boundary conditions for the δ'-interaction with some intensities in the isolated points of Γ. In this case, the number of negative eigenvalues of the operator $L_{\Gamma,\delta'}$ is not less than the number of isolated points of Γ having negative intensities of

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δ'-interaction (Theorem 1). If Γ is endowed with a Radon measure, then the Schrödinger operator with δ'-interaction on Γ can also be defined using boundary conditions on Γ (Theorem 2). We give constructive examples of Schrödinger operators with δ'-interactions on Γ having an infinite number of negative eigenvalues (Theorem 3). The classification of point interactions for a one–dimensional Schrödinger operator is briefly given in section 2. In section 8 we give the deficiency subspaces of the minimal operator so that it becomes possible to determine the set of all Schrödinger operators describing an interaction which takes place inside Γ.

2 Point interactions

The one-dimensional Schrödinger operator that describes a one-point interaction in a point x_0 is a self-adjoint operator on the space $L_2(R^1)$ and, for $x \neq x_0$, is given by the differential expression $-\frac{d^2}{dx^2}$. The maximal domain of the operator $-\frac{d^2}{dx^2}$ for $x \neq x_0$ is the Sobolev space $W_2^2(R^1 \setminus \{x_0\})$. For functions $\varphi, \psi \in W_2^2(R^1 \setminus \{x_0\})$, we have the Lagrange formula

$$(-\psi'', \varphi)_{L_2} - (\psi, -\varphi'')_{L_2} = \omega(\Gamma\psi, \Gamma\varphi), \tag{1}$$

where the boundary form ω is defined on the space E^4 of boundary values of the functions ψ and φ,

$$\Gamma\psi = \mathrm{col}\,(\psi(x_0 + 0), \psi(x_0 - 0), \psi'(x_0 + 0), \psi'(x_0 - 0)) \in E^4,$$

by the formula

$$\omega(\Gamma\psi, \Gamma\varphi) = \psi'(x_0 + 0)\bar\varphi(x_0 + 0) - \psi(x_0 + 0)\bar\varphi'(x_0 + 0)$$
$$- \psi'(x_0 - 0)\bar\varphi(x_0 - 0) + \psi(x_0 - 0)\bar\varphi'(x_0 - 0). \tag{2}$$

Self-adjoint restrictions of the maximal operator are defined by domains in terms of the corresponding boundary data that make a Lagrangian plane in the space E^4; it is a maximal subspace on which the boundary form satisfies $\varpi(\Gamma\psi, \Gamma\psi) = 0$. Since the boundary form (2) can be represented as

$$\omega(\Gamma\psi, \Gamma\varphi) = (\Gamma_1\psi, \Gamma_2\varphi)_{E^2} - (\Gamma_2\psi, \Gamma_1\varphi)_{E^2}, \tag{3}$$

where $\Gamma_1\psi = \mathrm{col}\,(\psi'(x_0+0), -\psi'(x_0-0))$, $\Gamma_2\psi = \mathrm{col}\,(\psi(x_0+0), \psi(x_0-0))$, the general self-adjoint boundary conditions are given by a unitary matrix U operating on the space E^2,

$$\Gamma_1\psi + i\Gamma_2\psi = U(\Gamma_1\psi - i\Gamma_2\psi). \tag{4}$$

The matrix U uniquely parametrizes the Lagrangian planes. This gives rise to a Schrödinger operator A_U on the space $L_2(R^1)$ with the domain consisting of all functions in the space $W_2^2(R^1 \setminus \{x_0\})$ satisfying boundary condition (4) and $A_U\psi = -\psi''(x)$, $x \neq x_0$. The Schrödinger operator A_U that describes a point interaction in the point x_0 is characterized with the matrix U.

Conditions (4) contain split boundary conditions of the form

$$\psi(x_0 + 0)\cos\alpha_+ - \psi'(x_0 + 0)\sin\alpha_+ = 0,$$
$$\psi(x_0 - 0)\cos\alpha_- - \psi'(x_0 - 0)\sin\alpha_- = 0, \tag{5}$$

where $\alpha_\pm \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. These boundary conditions define a non-transparent interaction in the point x_0. Conditions (5) correspond to a self-adjoint Schrödinger operator A on the space $L_2(R^1) = L_2(-\infty, x_0) \oplus L_2(x_0, +\infty)$. This operator can be decomposed into the direct sum $A = A_1 \oplus A_2$ of

self-adjoint operators A_1 and A_2 acting on the spaces $L_2(-\infty, x_0)$ and $L_2(x_0, +\infty)$ that correspond to boundary conditions (5) in the points $x = x_0 - 0$ and $x = x_0 + 0$, respectively.

A converse statement also holds true. If a self-adjoint Schrödinger operator A describes a one point interaction and admits a representation as a direct sum, $A = A_1 \oplus A_2$, then the functions in its domain satisfy boundary conditions (5) with some real numbers $\alpha_\pm$.

Boundary conditions (4) split if and only if the unitary matrix U is diagonal, $U = \text{diag}\left(e^{2i\alpha_+}, e^{-2i\alpha_-}\right)$. In this case, boundary conditions (4) are equivalent to conditions (5).

The one-dimensional Schrödinger operator corresponding to point interactions on a finite set $X = \{x_1, ..., x_n\}$ is a self-adjoint extension, to the space $L_2(R^1)$, of the minimal operator $L_{min,X}$ defined on the space $C_0^\infty(R^1 \setminus X)$ by $L_{min,X}\varphi(x) = -\varphi''(x)$ [3, 4]. All such self-adjoint extensions are described by Lagrangian planes in the Euclidean space E^{4n} of boundary data for the functions $\psi \in W_2^2(R^1 \setminus X)$. This leads to self-adjoint boundary conditions given by unitary matrices acting on E^{2n}. Localized self-adjoint boundary conditions have the form of (4) in every point $x_k \in X$, whereas localized indecomposable boundary conditions have the form [3]

$$\text{col}\left(\psi(x_k + 0), \psi'(x_k + 0)\right) = \Lambda_k\text{col}\left(\psi(x_k - 0), \psi'(x_k - 0)\right), \tag{6}$$

where the transmission matrices Λ_k can be written as $\Lambda_k = e^{i\eta_k}R_k$, where R_k is a real matrix, and $\det R_k = 1$, η_k is a real constant.

The boundary form (2) can be represented equivalently as

$$\omega(\Gamma\psi, \Gamma\varphi) = (\hat{\Gamma}_1\psi, \hat{\Gamma}_2\varphi)_{E_2} - (\hat{\Gamma}_2\psi, \hat{\Gamma}_2\varphi)_{E_2}, \tag{7}$$

where
$$\hat{\Gamma}_1\psi = \text{col}\left(\psi_s', \psi_s\right), \quad \hat{\Gamma}_2\psi = \text{col}\left(\psi_r, -\psi_r'\right), \tag{8}$$

$$\begin{aligned}
\psi_s &= \psi(x_0 + 0) - \psi(x_0 - 0); & \psi_s' &= \psi'(x_0 + 0) - \psi'(x_0 - 0); \\
\psi_r &= \tfrac{1}{2}[\psi(x_0 + 0) + \psi(x_0 - 0)]; & \psi_r' &= \tfrac{1}{2}[\psi'(x_0 + 0) + \psi'(x_0 - 0)].
\end{aligned} \tag{9}$$

By (7), general self-adjoint boundary conditions in the point x_0 are defined with a unitary matrix $\hat{U}$ acting on the space E^2 and have the form

$$\hat{\Gamma}_1\psi + i\Gamma_2\psi = \hat{U}(\hat{\Gamma}_1\psi - i\Gamma_2\psi). \tag{10}$$

The matrices $\hat{U}$ and U in the boundary conditions (4) and (10) are connected with each other via the relations
$$\hat{U} = (3CUC + 1)(3 + 3CUC)^{-1},$$
$$U = C^*(3 - \hat{U})(3\hat{U} - 1)C,$$

where $C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ is a unitary matrix.

Among one-point interactions, the following four cases are important.

1) The δ-interaction, or δ-potential, with intensity α is defined by the boundary conditions

$$\psi(x_0 + 0) - \psi(x_0 - 0) = 0, \qquad \psi'(x_0 + 0) - \psi'(x_0 - 0) = \alpha\psi_r(x_0), \tag{11}$$

where x_0 is the interaction point. In this case, the Λ-matrix in the boundary conditions (6) has the form $\Lambda = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$.

2) The δ'-interaction with intensity β is defined by the boundary conditions

$$\psi'(x_0 + 0) - \psi'(x_0 - 0) = 0, \qquad \psi(x_0 + 0) - \psi(x_0 - 0) = \beta\psi'_r(x_0). \qquad (12)$$

In this case, the Λ-matrix in the boundary conditions (6) has the form $\Lambda = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$.

3) The δ'-potential with intensity γ is defined with the boundary conditions

$$\psi(x_0 + 0) - \psi(x_0 - 0) = \gamma\psi_r(x_0), \qquad \psi'(x_0 + 0) - \psi'(x_0 - 0) = -\gamma\psi'_r(x_0). \qquad (13)$$

An equivalent form of the boundary conditions (13) is $\psi(x_0 + 0) = \theta\psi(x_0 - 0)$, $\psi'(x_0 + 0) = \theta^{-1}\psi'(x_0 - 0)$, where $\theta = \dfrac{2 + \gamma}{2 - \gamma}$. In this case, the matrix Λ in the boundary conditions (6) is $\Lambda = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$.

4) The δ-magnetic potential with intensity μ is defined in terms of the boundary conditions

$$\psi(x_0 + 0) - \psi(x_0 - 0) = i\mu\psi_r(x_0), \quad \psi'(x_0 + 0) - \psi'(x_0 - 0) = i\mu\psi'_r(x_0), \qquad (14)$$

where i is the imaginary unit. An equivalent form of the boundary conditions (14) is $\psi(x_0 + 0) = e^{i\eta}\psi(x_0 - 0)$, $\psi'(x_0 + 0) = e^{i\eta}\psi'(x_0 - 0)$, where $\frac{\mu}{2} = \tan\frac{\eta}{2}$. In this case, Λ in the boundary conditions (6) is a multiple of the identity matrix, $\Lambda = e^{i\eta}I$.

To explain the names and the physical meaning of the four types of interactions listed above, consider at first the formal Schrödinger operators L,

$$L = -\frac{d^2}{dx^2} + \varepsilon\delta^{(j)}(x - x_0), \ j = 0, 1; \ \varepsilon = \alpha, \ j = 0; \ \varepsilon = \gamma, \ j = 1, \qquad (15)$$

the expression $L\psi$ can be defined in the sense of distribution theory for functions $\psi \in W_2^2(R^1 \setminus \{x_0\})$.

Indeed, the expression $-\frac{d^2}{dx^2}$ on such functions ψ, in the sense of distribution theory, is given by the expression

$$-\frac{d^2}{dx^2}\psi(x) = -\psi''(x) - \delta'(x - x_0)\psi_s(x_0) - \delta(x - x_0)\psi'_s(x_0). \qquad (16)$$

The product $\delta^{(j)}(x - x_0)\psi(x)$ is well defined if $\psi \in C^\infty(R^1)$, that is, the function ψ is a multiplicator for the Schwartz space $C_0^\infty(R^1)$ of test functions. In this case,

$$\delta(x - x_0)\psi(x) = \psi(x_0)\delta(x - x_0), \ \delta'(x - x_0)\psi(x) = \psi_r(x_0)\delta'(x - x_0) - \psi'_r(x_0)\delta(x - x_0). \qquad (17)$$

The identity (17) can be extended as to also encompass discontinuous functions $\psi \in C^\infty(R^1 \setminus \{x_0\})$ by defining the functionals $\delta^{(j)}(x - x_0)$ by $(\delta^{(j)}(x - x_0), \varphi(x)) = (-1)^j\varphi_r^{(j)}(x_0)$ [4]. Hence, with such a definition, formulas (17) hold if all $\psi^{(j)}(x_0)$ in the right-hand sides of formulas (17) are replaced with $\psi_r^{(j)}(x_0)$.

If (16) and (17) are used in (15), then the condition $L\psi \in L_2(R^1)$ leads to (11) if $j = 0$ and to (13) if $j = 1$.

Consider now a one-dimensional Schrödinger operator with magnetic field potential a and potential V, that is, $L = (i\frac{d}{dx} + a)^2 + V$, in the particular case where $L = -\frac{d^2}{dx^2} + 2ia\frac{d}{dx} + ia'$ and $a(x) = \mu\delta(x)$, so that

$$L_\mu\psi = -\frac{d^2\psi}{dx^2} + 2ia\delta(x)\frac{d\psi}{dx} + i\mu\delta'(x)\psi(x). \tag{18}$$

If we use expressions (16), (17) in (18), then imposing the condition on $\psi(x) \in W_2^2(R^1 \setminus \{x_0\})$ that the distribution $L_\mu\psi$ is a usual function in $L_2(R^1)$ leads to (14). Hence, the boundary conditions (14) describe a magnetic field with the potential $a(x) = \mu\delta(x)$.

Particular forms of the boundary conditions (11)–(14) can be represented as

$$\left(\begin{array}{c} \psi_s'(x_0) \\ \psi_s(x_0) \end{array} \right) = B \left(\begin{array}{c} \psi_r(x_0) \\ -\psi_r'(x_0) \end{array} \right), \tag{19}$$

where ψ_s, ψ_s', ψ_r, and ψ_r' are defined in (9). The matrix $B = \left(\begin{array}{cc} \alpha & \gamma - i\mu \\ \gamma + i\mu & -\beta \end{array} \right)$ is self-adjoint and each condition in (10)–(14) follows from (19) by setting three of the four parameters α, β, γ, μ to zero. For an arbitrary self-adjoint matrix B, the conditions (19) make a particular case of self-adjoint boundary conditions of the form (10) with the unitary matrix $\hat{U} = (B - i)^{-1}(B + i)$.

Note that the boundary conditions (19) do not contain all non-splitting self-adjoint boundary conditions of the form (4). In particular, they do not include boundary conditions of the form

$$\psi'(x_0 + 0) = i\lambda_0\psi(x_0 - 0), \quad \psi'(x_0 - 0) = i\lambda_0\psi(x_0 + 0) \tag{20}$$

with a real constant λ_0. The boundary conditions (20) describe a point interaction, in the point $x = x_0$, transparent for the waves $e^{i\lambda x}$ with $\lambda = \lambda_0$. In this case, the function $\psi = e^{i\lambda_0 x}$ satisfies the boundary conditions (20) and the Schrödinger equation. Boundary conditions (20) have the form (6) with the matrix $\Lambda = i \left(\begin{array}{cc} 0 & -\lambda_0^{-1} \\ \lambda_0 & 0 \end{array} \right)$.

Let us also give a relation between the matrix Λ from the boundary condition (6) and the matrix B from the conditions (19),

$$\Lambda = \frac{1}{D} \left\| \begin{array}{cc} \theta_+ & \beta \\ \alpha & \theta_- \end{array} \right\|,$$

where $D = (1 - \frac{i}{2}\mu)^2 - \frac{1}{4}\alpha\beta - \frac{1}{4}\gamma^2$, $\theta_\pm = (1 \pm \frac{\gamma}{2})^2 + \frac{1}{4}\alpha\beta + \frac{1}{4}\mu^2$.

The Schrödinger operator L_B corresponding to the boundary conditions (19) for a point interaction in the point $x_0 = 0$ can formally be represented with the following expression containing the Dirac δ-function and its derivative $\delta'(x)$,

$$L_B = -\frac{d^2}{dx^2} + \alpha\delta(x)(\cdot, \delta) - \beta\delta'(x)(\cdot, \delta') + (\gamma + i\mu)\delta'(x)(\cdot, \delta) + (\gamma - i\mu)\delta(x)(\cdot, \delta'). \tag{21}$$

Here the differentiation $\frac{d^2}{dx^2}$ is understood in the distribution sense, and the functionals $(\cdot, \delta)$ and $(\cdot, \delta')$ are defined by $(\psi, \delta) = \psi_r(0) = \frac{1}{2}[\psi(+0) + \psi(-0)]$, $(\psi, \delta') = -\psi_r'(0) = -\frac{1}{2}[\psi'(+0) + \psi'(-0)]$. The domain of the operator L_B is defined by the condition $L_B\psi \in L_2(R^1)$ imposed on the functions ψ [4].

It is well known [3,4] that a model for point interactions is exactly solvable and can serve as a good approximation of real Schrödinger operators if the potential v has small support in

a neighborhood of the point x_0, that is, $v(x) = 0$ for $|x - x_0| > \varepsilon$, and the processes under the study have the energy λ^2 much less than ε^{-2}. Here it is assumed that, for the energies under consideration, the matrix Λ_ε that connects values of solutions ψ of the Schrödinger equation $[-\frac{d^2}{dx^2} + v]\psi = \lambda^2\psi$ and their derivatives $\psi'(x)$ for $x = x_0 - \varepsilon$ and $x_0 + \varepsilon$, that is, $\mathrm{col}\,(\psi(x_0 + \varepsilon), \psi'(x_0 + \varepsilon)) = \Lambda_\varepsilon \mathrm{col}\,(\psi(x_0 - \varepsilon), \psi'(x_0 - \varepsilon))$, is close to the matrix Λ that defines the boundary conditions (6) for the point interaction. Thus the Schrödinger operator with point interaction can be considered as a limit (in a certain sense, e.g., in the sense of uniform resolvent convergence), as $\varepsilon \to 0$, of Schrödinger operators with the potentials $v_\varepsilon(x)$ with $\Lambda_\varepsilon \to \Lambda$ for $\varepsilon \to 0$. Here, the potentials $v_\varepsilon(x)$ themselves may or may not have a limit as $\varepsilon \to 0$ even in the sense of distributions. It can happen that their limit values, even if they exist, do not determine the character and the intensity of the point interaction.

Let us look at this phenomenon in greater details for the case of δ'-potentials; this case was considered in a number of papers [2, 16-19,21,28,29,31-35]. For a model of δ-potentials with intensity α, one can take a sequence of regular potentials $v_\varepsilon(x) \to \alpha\delta(x)$ with $\varepsilon \to 0$, for example, $v_\varepsilon(x) = \alpha\varepsilon^{-1}\varphi(\frac{x}{\varepsilon})$, where the compactly supported function φ is such that $\int \varphi(x)\,dx = 1$. More complex potentials can be well modeled on small intervals by a sum of several δ-functions,

$$v_\varepsilon(x) = \sum_{j=1}^{N} \alpha_j(\varepsilon)\delta(x - x_j(\varepsilon)), \tag{22}$$

where all $x_j(\varepsilon) \to x_0$ for $\varepsilon \to 0$. It is shown in [6] that the δ'-interaction is well modeled with three approaching δ-functions that have special opposite sign increasing intensities $\alpha_j(\varepsilon)$. When modeling a δ'-potential of intensity γ, the number of terms in representation (22) depends on the conditions to be satisfied. Since the matrix Λ in the boundary conditions (6) is diagonal for the δ'-potential of intensity γ, there are two necessary conditions on the elements of the matrix Λ_ε,

1) $\lim\limits_{\varepsilon \to 0} (\Lambda_\varepsilon)_{2,1} = 0$,

2) $\lim\limits_{\varepsilon \to 0} (\Lambda_\varepsilon)_{1,1} = (1 + \frac{\gamma}{2})(1 - \frac{\gamma}{2})^{-1}$.

These two conditions can be satisfied with two terms in approximation (22),

$$v_\varepsilon(x) = \alpha_1\varepsilon^{-1}\delta(x) + \alpha_2\varepsilon^{-1}\delta(x - \varepsilon), \tag{23}$$

where $\alpha_1 = \gamma(1 - \frac{\gamma}{2})^{-1}$, $\alpha_2 = -\gamma(1 + \frac{\gamma}{2})^{-1}$.

Here, the potentials v_ε do not have a limit as $\varepsilon \to 0$ in the sense of distributions. In this case, the matrix Λ_ε can be written as a product of three matrices $\Lambda_\varepsilon = \Lambda_2\Lambda_\varepsilon^0\Lambda_1$, where $\Lambda_j = \begin{pmatrix} 1 & 0 \\ \alpha_j\varepsilon^{-1} & 1 \end{pmatrix}$, $j = 1, 2$, $\Lambda_\varepsilon^0 = \begin{pmatrix} \cos\lambda\varepsilon & \frac{\sin\lambda\varepsilon}{\lambda} \\ -\lambda\sin\lambda\varepsilon & \cos\lambda\varepsilon \end{pmatrix}$. These matrices give a relation between the solutions $\psi(x)$ of the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + v_\varepsilon\psi = \lambda^2\psi$$

and its derivatives $\psi'(x)$ in different points x,

$$\mathrm{col}\,(\psi(+0), \psi'(+0)) = \Lambda_1\mathrm{col}\,(\psi(-0), \psi'(-0)),$$
$$\mathrm{col}\,(\psi(\varepsilon - 0), \psi'(\varepsilon - 0)) = \Lambda_\varepsilon^0\mathrm{col}\,(\psi(+0), \psi'(+0)),$$
$$\mathrm{col}\,(\psi(\varepsilon + 0), \psi'(\varepsilon + 0)) = \Lambda_2\mathrm{col}\,(\psi(\varepsilon - 0), \psi'(\varepsilon - 0)).$$

Using the explicit form of α_j we get

$$\lim_{\varepsilon \to 0} \Lambda_\varepsilon = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix},$$

where $\theta = \dfrac{2+\gamma}{2-\gamma}$. Hence, the limit Schrödinger operator corresponds to a point interaction having δ'-potential of intensity γ.

One can additionally require that $v_\varepsilon(x) \to \kappa \delta'(x)$ in (22) as $\varepsilon \to 0$. This can be achieved if we take

$$v_\varepsilon(x) = \alpha_1 \varepsilon^{-1} \delta(x + \varepsilon) + \alpha_2 \varepsilon^{-1} \delta(x) + \alpha_3 \varepsilon^{-1} \delta(x - \varepsilon) \tag{24}$$

in (22), where $\alpha_2 = \pm 2\gamma[\gamma^2 - 4]^{-\frac{1}{2}}$, $\alpha_1 = \frac{\gamma}{2} + \frac{\alpha_2}{2}(1 + \frac{\gamma}{2})$, $\alpha_3 = -\frac{\gamma}{2} - \frac{\alpha_2}{2}(1 - \frac{\gamma}{2})$.

In the limit as $\varepsilon \to 0$, the Schrödinger operators with the potentials $v_\varepsilon(x)$ of the form (24) define point interaction of δ'-potential type with intensity γ, and the limit $v_\varepsilon(x) \to \kappa \delta'(x)$ exists in the distribution sense, where the constant $\kappa = \alpha_1 - \alpha_3 = \gamma + \alpha_2$ depends on the choice of the sign of α_2 and, consequently, it does not determine the intensity γ. Moreover, considering an expression of the form (22) for the potentials $v_\varepsilon(x)$ with four terms

$$v_\varepsilon(x) = \alpha_1 \varepsilon^{-1} \delta(x) + \alpha_2 \varepsilon^{-1} \delta(x - \varepsilon) + \alpha_3 \varepsilon^{-1} \delta(x - 2\varepsilon) + \alpha_4 \varepsilon^{-1} \delta(x - 3\varepsilon), \tag{25}$$

where $\alpha_1 = -1$, $\alpha_2 = 6$, $\alpha_3 = -3$, $\alpha_4 = -2$ we obtain $\lim_{\varepsilon \to 0} v_\varepsilon(x) = 6\delta'(x)$ in the sense of distributions. On the other hand, it is easy to see that $\lim_{\varepsilon \to 0} \Lambda_{3\varepsilon} = I$, that is, if $\varepsilon \to 0$, the Schrödinger operators with potentials (25) converge to a free Schrödinger operator. By taking $\alpha_1 = \alpha_4 = 3$, $\alpha_2 = \alpha_3 = -3$ in (25), we have $v_\varepsilon(x) \to 0$ and the Schrödinger operators converge to a direct sum of operators on the spaces $L_2(-\infty, 0)$ and $L_2(0, +\infty)$ corresponding to the Dirichlet conditions $\psi(\pm 0) = 0$.

Let us remark that if the Schrödinger operators have potentials in the form of (23)—(25), then the kernels of the resolvents for these operators can be written explicitly similarly to the case of the limit Schrödinger operator. This yields that these operators converge, as $\varepsilon \to 0$, in the sense of uniform resolvent convergence.

The above conclusions about Schrödinger operators with potentials (23)– (25) remain also true if v_ε are piecewise constant or even $v_\varepsilon \in C_0^\infty(R^1)$ if they can well approximate each term in (23)—(25).

Let us also make a remark on one more feature of point interactions. If the support of the potential $v_\varepsilon(x)$ belongs to the interval $(-\varepsilon, \varepsilon)$ and its components $v_\varepsilon^-(x) = \theta(-x)v_\varepsilon(x)$, $v_\varepsilon^+(x) = \theta(x)v_\varepsilon(x)$, where θ is the unit Heaviside function, determine point interactions with the corresponding matrices Λ^- and Λ^+, as $\varepsilon \to 0$, then the potential $v_\varepsilon(x)$ also gives rise to a point interaction, as $\varepsilon \to 0$, with the matrix $\Lambda = \Lambda^+ \Lambda^-$. This leads to additivity of intensities α and β for δ- and δ'-interactions, since they correspond to triangular matrices Λ^-, Λ^+, Λ. For point interactions with δ'-type potentials and δ-magnetic potentials, the intensities γ and μ do not have such an additivity property. Here, if γ_- and γ_+ are intensities of δ'-potentials corresponding to v_ε^- and v_ε^+, then the total intensity γ is found as $\gamma = (\gamma_- + \gamma_+)(1 + \frac{1}{4}\gamma_-\gamma_+)^{-1}$. Thus, for point interactions with δ'-type potential and δ-magnetic potential, the "additive" characteristics of the intensities are useful. The additive characteristic ξ for δ'-potential with intensity γ are defined by the identities $\frac{2+\gamma}{2-\gamma} = \pm e^{\xi \pm}$, where the sign "+" is taken if $|\gamma| < 2$ and we take the sign "−" if $|\gamma| > 2$. A more exact definition of additive characteristic for point interactions with δ'-potential is the following. Additive characteristic is a pair (ξ, s) consisting of the number ξ and the sign $s = \pm 1$. As two-point interactions with δ'-potentials having characteristics (ξ_1, s_1)

and (ξ_2, s_2) approach, the total characteristic (ξ, s) is found as $(\xi, s) = (\xi_1 + \xi_2, s_1 \cdot s_2)$, which corresponds to the above "adding" rule for the intensities γ_- and γ_+.

For a point interaction with δ-magnetic potential of intensity μ, the Λ-matrix in the boundary condition (6) is a multiple of the identity matrix, $\Lambda = e^{i\eta}I$. Hence, it is convenient to take the number η to be an "additive" characteristic of the δ-magnetic potential. There is a relation between μ and η, $\mu = 2\tan\frac{\eta}{2}$. For two approaching point interactions with δ-magnetic potentials having characteristics η_1 and η_2, the corresponding total characteristic is $\eta = \eta_1 + \eta_2$.

It is not true that if the Schrödinger operators $-\frac{d^2}{dx^2} + v_\varepsilon(x)$ converge, as $\varepsilon \to 0$, to a Schrödinger operator with point interaction of a certain type then the operators $-\frac{d^2}{dx^2} + kv_\varepsilon(x)$, where $k \neq 1$ is an arbitrary real constant, also converge to a Schrödinger operator with point interaction of the same type. In the general case, this is true only for δ-potential. It is shown in [21] that, for special approximations of $\alpha\delta'$-functions where $v_\varepsilon = \alpha\varepsilon^{-2}\psi(\frac{x}{\varepsilon})$, $\int \psi(x)\,dx = 0$, $\int x\psi(x)\,dx = -1$, the Schrödinger operators have a limit that defines a point interaction of δ'-potential only for special "resonance" values of α.

Proposition 1. *For a one-dimensional Schrödinger operator A with local interactions on a finite set $X = \{x_1, ..., x_n\}$, to describe a δ'-interaction it is necessary and sufficient that all the functions $\chi(x) \in C_0^\infty(R^1)$ such that $\chi'(x) \in C_0^\infty(R^1 \setminus X)$ belong to the domain of the operator A and the operator A does not admit a representation as a direct sum $A = A_1 \oplus A_2$ of two self-adjoint operators on the spaces $L_2(-\infty, a)$ and $L_2(a, +\infty)$ for any a.*

Proof. Necessity follows, since the boundary conditions for a δ'-interaction can not be represented in the form (5), that is, the operator with δ'-interaction can not be represented as $A = A_1 \oplus A_2$. Moreover, each function $\chi(x) \in C_0^\infty(R^1)$ satisfying $\chi'(x) \in C_0^\infty(R^1 \setminus X)$ assumes constant values in small neighborhoods of the points $x_k \in X$. Hence, this function satisfies the boundary conditions for δ'-interaction on the set X with arbitrary intensities.

Sufficiency follows, since if the operator A does not admit the representation $A = A_1 \oplus A_2$ on the space $L_2(-\infty, a) \oplus L_2(a, +\infty)$ and the function $\chi(x) \in \mathfrak{D}(A)$ is distinct from zero only in a small neighborhood of the point x_k not containing other points of X, the boundary condition (6) leads to the matrix $\Lambda = \begin{pmatrix} 1 & \beta_k \\ 0 & 1 \end{pmatrix}$ with real β_k. This corresponds to δ'-interaction in the point x_k with intensity β_k. $\square$

3 Interactions on a set of measure zero

Let Γ be a closed bounded subset of R^1 of Lebesgue measure zero, $|\Gamma| = 0$. There is a symmetric minimal operator $L_{min,\Gamma}$ defined on the space $L_2(R^1)$ by $L_{min,\Gamma}\varphi(x) = -\varphi''(x)$ on functions $\varphi \in C_0^\infty(R^1 \setminus \Gamma)$. An operator adjoint in $L_2(R^1)$ to the operator $L_{min,\Gamma}$ is maximal. Its domain is $\mathfrak{D}(L_{\max,\Gamma}) = W_2^2(R^1 \setminus \Gamma)$.

Each self-adjoint operator A, that is, a self-adjoint extension of the operator $L_{min,\Gamma}$, defines an interaction on the set Γ.

Definition 1. We will say that a self-adjoint operator $A \supset L_{min,\Gamma}$ defines a local interaction on Γ if $u(x) \in \mathfrak{D}(A)$ implies that $\chi(x)u(x) \in \mathfrak{D}(A)$ for an arbitrary cutting function $\chi(x) \in C_0^\infty(R^1)$ such that $\chi'(x) \in C_0^\infty(R^1 \setminus \Gamma)$. In this case, we also say that the functions in $\mathfrak{D}(A)$ satisfy local boundary conditions.

Lemma 1. *Let A_Γ be a self-adjoint operator on the space $L_2(R^1)$ describing a local interaction on Γ. Let $a, b \notin \Gamma$, $a < b$. Then, the second Green formula holds true for any functions $f, g \in \mathfrak{D}(A_\Gamma)$*

$$\int\limits_a^b [(A_\Gamma f)(x)\overline{g(x)} - f(x)\overline{(A_\Gamma g)(x)}]\, dx = f(b)\overline{g'(b)} - f'(b)\overline{g(b)} - f(a)\overline{g'(a)} + f'(a)\overline{g(a)}. \quad (26)$$

In case of $a = -\infty$ or $b = +\infty$ in the rightside of (26) there are no terms of boundary data of functions f, g at points $a = -\infty$ or $b = +\infty$.

Proof. Let $a_- < a$ and $b_+ > b$ be such that the intervals (a_-, a) and (b, b_+) contain no points of Γ. Let $\varphi(x) \in C_0^\infty(a_-, b_+)$ be a cutting function that equals to 1 with $x \in (a, b)$. The functions $f_0 = \varphi \cdot f$ and $g_0 = \varphi \cdot g$ belong to domain of the operator A_Γ since the operator A_Γ describes a local interaction on Γ according to definition 1. The functions f_0, g_0, $A_\Gamma f_0$, $A_\Gamma g_0$ coincide with f, g, $A_\Gamma f$, $A_\Gamma g$ at $x \in (a, b)$, respectively. Therefore the righthandside of (26) can be written in the following form:

$$\int\limits_a^b [(A_\Gamma f_0)(x)\overline{g_0(x)} - f_0(x)\overline{(A_\Gamma g_0)(x)}]\, dx =$$

$$\int\limits_{a_-}^{b_+} [(A_\Gamma f_0)(x)\overline{g_0(x)} - f_0(x)\overline{(A_\Gamma g_0)(x)}]\, dx - \int\limits_{(a_-,a)\cup(b,b_+)} [-f_0''\overline{g_0} + f_0\overline{g_0''}]\, dx =$$

$$(A_\Gamma f_0, g_0) - (f_0, A_\Gamma g_0) + f_0(b)\overline{g_0'(b)} - f_0'(b)\overline{g_0(b)} - f_0(a)\overline{g_0'(a)} + f_0'(a)\overline{g_0(a)}.$$

This leads to equality (26) since the operator A_Γ is a self–adjoint on $L_2(R^1)$ and the functions f_0, g_0 coincide with f, g on the interval $[a, b]$.

In case of $a = -\infty$, taking of $a_- = -\infty$, we do not have terms with boundary data of functions f and g at the point $a = -\infty$. In the same way, in the case when $b = +\infty$, we do not have terms of values f and g at the point $b = +\infty$. Let us note that the boundary data $\Gamma_f = (f(a), f(a)', f(b), f(b)')$ of functions $f \in \mathfrak{D}(A_\Gamma)$ fill all the space E^4. $\qquad\square$

Proposition 2. *Let the conditions of Lemma 1 hold. Let $A_\Gamma^{(a,b)}$ be a restriction of the operator A_Γ on the space $L_2(a, b)$ acting as following $A_\Gamma^{(a,b)}[\chi_{(a,b)}f] = \chi_{(a,b)}A_\Gamma f$ for any function $f \in \mathfrak{D}(A_\Gamma)$ such that $f(a) = f(b) = 0$. Here, $\chi_{(a,b)}(x)$ is a characteristic function on the interval (a, b), i.e. $\chi_{(a,b)}(x) = 1$ with $x \in (a, b)$, and $\chi_{(a,b)}(x) = 0$ with $x \notin (a, b)$. Then, $A_\Gamma^{(a,b)}$ is a self–adjoint operator on the $L_2(a, b)$.*

Proof. The definition of the operator $A_\Gamma^{(a,b)}$ is correct, since the operator A_Γ is a local operator $A_\Gamma \psi(x) = -\psi''(x), x \notin \Gamma$. Formula (26) shows that the operator $A_\Gamma^{(a,b)}$ is a symmetric restriction of maximal operator for functions that correspond to self–adjoint boundary conditions $f(a) = f(b) = 0$. Therefore, $A_\Gamma^{(a,b)}$ is a self–adjoint operator on $L_2(a, b)$. $\qquad\square$

Lemma 2. *Let a self-adjoint operator A_Γ define a local interaction on Γ. Let $x_0 \in \Gamma$ be an isolated point of the set Γ. Then the functions in $\mathfrak{D}(A_\Gamma)$ satisfy local boundary conditions (4) in the point x_0.*

Proof. Let $x_0 \in \Gamma$ be an isolated point of the set Γ. Then there exists an interval (a, b) such that (a, b) contains no other points of the set Γ except for x_0. Let us consider all functions $\psi, \varphi \in \mathfrak{D}(A_\Gamma)$ that equal to zero at $x = a, b$. Then,

$$(A_\Gamma^{(a,b)}\psi, \varphi) - (\psi, (A_\Gamma^{(a,b)}\varphi) = \int\limits_a^b [-\psi''(x)\overline{\varphi(x)} - \psi(x)\overline{\varphi''(x)}]\, dx = \omega(\Gamma\psi, \Gamma\varphi),$$

9

where the boundary form ω is defined in (2). Since the operator $A_\Gamma^{(a,b)}$ is a self–adjoint operator on $L_2(a,b)$ in virtue of Proposition 2 then functions $\psi \in \mathfrak{D}(A_\Gamma^{(a,b)})$ must satisfy the boundary condition (4).

$\square$

Definition 2. We will say that a self-adjoint operator A admits splitting boundary conditions in a point $x_0 \in \Gamma$ if the operator A on the space $L_2(R^1) = L_2(-\infty, x_0) \oplus L_2(x_0, +\infty)$ admits a representation in the form of the direct sum $A = A_1 \oplus A_2$ of a self-adjoint operator A_1 on the space $L_2(-\infty, x_0)$ and an operator A_2 on the space $L_2(x_0, +\infty)$.

Lemma 3. *Let $x_0 \in \Gamma$ be an isolated point of the set Γ. Let a self-adjoint operator A_Γ define a local interaction on Γ and not admit splitting boundary conditions in a point x_0. Then the functions in $\mathfrak{D}(A)$ satisfy non-splitting boundary conditions in the point x_0 of the form (6).*

Proof. The proof follows from Lemma 2 and the general form of (6) for non-splitting boundary conditions. $\square$

Definition 3. We say that a self-adjoint operator $A \supset L_{min,\Gamma}$ describes a δ'-interaction on Γ if the operator A corresponds to local non-splitting boundary conditions on Γ and all the functions $\chi(x) \in C_0^\infty(R^1)$ satisfying $\chi'(x) \in C_0^\infty(R^1 \setminus \Gamma)$ belong to $\mathfrak{D}(A)$.

Lemma 4. *Let a self-adjoint operator $A \supset L_{min,\Gamma}$ define a δ'-interaction on Γ. If $x_0 \in \Gamma$ is an isolated point of the set Γ, then there exists a real number β such that the functions in $\mathfrak{D}(A)$ satisfy boundary conditions (12) for point δ'-interactions with intensity β.*

Proof. The proof follows since a function $\varphi_0(x)$, that equals 1 on a small neighborhood of the point x_0, belongs to $\mathfrak{D}(A)$, and satisfies the local non-splitting boundary condition (6) if and only if the boundary condition describes a δ'-interaction (see the proof of Proposition 1). $\square$

4   Test functions for δ'−interactions

Let $x_0 \in \Gamma$ be an isolated point of a bounded closed set Γ having Lebesgue measure zero. Let a self-adjoint operator A define a δ'-interaction on Γ. In particular, the functions $\psi(x) \in \mathfrak{D}(A)$, in the point x_0, satisfy the boundary conditions

$$\psi'(x_0 + 0) = \psi'(x_0 - 0), \qquad \psi(x_0 + 0) - \psi(x_0 - 0) = \beta \frac{1}{2}[\psi'(x_0 + 0) + \psi'(x_0 - 0)] \qquad (27)$$

where β is intensity of the δ'-interaction in the point x_0. We construct a function that belongs to $\mathfrak{D}(A)$, has compact support, satisfies condition (27) in the point x_0, and consists piecewise of parabolas and constants.

Definition 4. Consider the following test function that depends on 4 parameters ε, β, l, r:

$$t(x, \varepsilon, \beta, l, r) = \begin{cases} 0, & x \le -\varepsilon, \\ \frac{1}{2\varepsilon}(x + \varepsilon)^2, & -\varepsilon \le x < 0, \\ \beta + \varepsilon - \frac{1}{2\varepsilon}(x - \varepsilon)^2, & 0 < x \le \varepsilon, \\ \beta + \varepsilon, & \varepsilon \le x \le l, \\ \beta + \varepsilon - \frac{\beta + \varepsilon}{2r^2}(l - x)^2, & l \le x \le l + r, \\ \frac{\beta + \varepsilon}{2r^2}(l + 2r - x)^2, & l + r \le x \le l + 2r, \\ 0, & l + 2r \le x. \end{cases}$$

Proposition 3. *The following 4 properties of test functions easily follow from Definition 4.*

1) *A test function is twice (weakly) differentiable with compact support for $x \neq 0$.*

2) *The test function $\hat{t}(x) = t(x - x_0; \varepsilon, \beta, l, r)$ satisfies condition (27) for a δ'-interaction with intensity β.*

3) *If $0 < \varepsilon \leq \varepsilon_0$ is such that the $2\varepsilon_0$-neighborhood of the point x_0 contains no points of Γ other than x_0 and the value of l is larger than the diameter of Γ, then the test function $\hat{t}(x) = t(x - x_0; \varepsilon, \beta, l, r)$ belongs to the domain of any self-adjoint operator A defining a δ'-interaction on Γ and the δ'-interaction in the point x_0 with intensity β.*

4) *If 3) holds, then $(A\hat{t}, \hat{t}) = \beta + \dfrac{2}{3}\varepsilon + \dfrac{2}{3r}(\beta + \varepsilon)^2$.*

5 Number of negative eigenvalues for δ'-interaction

It is well known [7, 23] that for point δ'-interactions at finitely many points the number of negative eigenvalues of the Schrödinger operator equals the number of points having negative intensities of the δ'-interactions. In the case of infinitely many points it can happen that the point spectrum is empty, cf. [3], Theorem 3.6. However, for bounded Γ and if the negative spectrum is discrete, there is the following generalization of the mentioned result on the number of negative eigenvalues.

Theorem 1. *Let $A_{\Gamma, \delta'}$ be a self-adjoint Schrödinger operator on $L_2(R^1)$ with δ'-interaction on a closed bounded set Γ of Lebesgue measure zero. Let the negative spectrum of the operator $A_{\Gamma, \delta'}$ be discrete. Then the number of negative eigenvalues of the operator $A_{\Gamma, \delta'}$ is not less than the number of isolated points of the set Γ having negative values of intensities of the δ'-interactions.*

Proof. Let $x_1, ..., x_n$ be isolated points of the set Γ having negative values $\beta_k < 0$ of intensities of δ'-interactions in the points x_k, $k = 1, ..., n$. Let $\varepsilon_0 > 0$ be a sufficiently small number such that the $2\varepsilon_0$-neighborhood of each point $x_k \in \Gamma$ contains no points of the set Γ other than x_k. Let $\mathcal{L}_n$ be an n-dimensional subspace of $\mathfrak{D}(A)$ containing the test-functions $\hat{t}(x) = t(x - x_k; \beta_k, \varepsilon_k, l_k, r_k)$, $k = 1, ..., n$, corresponding to the points x_k with the intensities $\beta_k < 0$. Choose numbers $\varepsilon_k \leq \varepsilon_0$ and r_k such that

$$\beta_k + \frac{2}{3}\varepsilon_k + \frac{2}{3r_k}(\beta_k + \varepsilon_k)^2 = \frac{1}{2}\beta_k < 0. \tag{28}$$

Moreover, choose all $l_k \geq l$, where l is larger than the diameter of Γ, and such that the intervals $I_k = (l_k, l_k + 2r_k)$, $k = 1, ..., n$, do not intersect for distinct k. Hence, every function $u \in \mathcal{L}_n$ can be represented as

$$u(x) = \sum_{k=1}^{n} a_k t_k(x - x_k; \beta_k, \varepsilon_k, l_k, r_k), \tag{29}$$

where a_k are complex constants. Using properties of test functions and (28) it is easy to see that the quadratic form (Au, u) is negative definite on the n-dimensional subspace $\mathcal{L}_n$, i.e. for $u \in \mathcal{L}_n \setminus \{0\}$ we have

$$(Au, u) = \sum_{k=1}^{n} |a_k|^2 (At_k, t_k) = \frac{1}{2} \sum_{k=1}^{n} \beta_k |a_k|^2 < 0. \tag{30}$$

Hence, it follows from the variational minimax principle [26] that the operator A has at least n negative eigenvalues. $\square$

6 Boundary conditions for δ'-interactions

If $\Gamma = X$ is a finite or countable set of points, $X = \{x_k\}_{k=1}^{\infty}$, then the Schrödinger operator $L_{X,\beta}$ with δ'-interaction in the points $x_k \in X$ with intensities β_k is defined on functions that belong to the space $W_2^2(R^1 \setminus X)$ and satisfy the boundary conditions (12) in every point $x = x_k$.

Let Γ be a closed bounded subset of R^1 of measure zero, $|\Gamma| = 0$. The Schrödinger operator with δ'-interaction on Γ is defined in an abstract form in Section 3 (see Definition 3). We will give a concrete construction of such operators following [8, 24].

Let Γ be endowed with a Radon measure, that is, a finite regular Borel measure μ [30] such that its support coincides with Γ. In this case, one can define boundary data on Γ for some functions $\psi \in W_2^2(R^1 \setminus \Gamma)$, which is an analogue of $\psi_s(x_0)$, $\psi_s'(x_0)$, $\psi_r(x_0)$, $\psi_r'(x_0)$ given in (9).

Let a function $\psi(x)$ and its derivative $\psi'(x)$ have the following representations for $x, s \in R^1 \setminus \Gamma$:

$$
\begin{aligned}
\psi(x) &= \psi(s) + \int_s^x \psi'(\xi) \, d\xi + \int_{(s,x)} f(\xi)\mu(d\xi), \\
\psi'(x) &= \psi'(s) + \int_s^x \psi''(\xi) \, d\xi + \int_{(s,x)} g(\xi)\mu(d\xi),
\end{aligned}
\tag{31}
$$

where f and g are defined on Γ and absolutely integrable with respect to the measure μ. The functions f and g are called derivatives of the functions $\psi(x)$ and $\psi'(x)$ with respect to the measure μ, and are denoted by $f = \frac{d\psi}{d\mu}$, $g = \frac{d\psi'}{d\mu}$. They are analogues of the jump functions $\psi_s(x_0)$ and $\psi_s'(x_0)$. It follows from (31) that there exist functions $\psi_r(x) = \frac{1}{2}[\psi(x+0) + \psi(x-0)]$ and $\psi_r'(x) = \frac{1}{2}[\psi'(x+0) + \psi'(x-0)]$ on Γ that are essentially bounded on Γ, i.e., belong to the space $L_\infty(\Gamma, d\mu)$. All four functions ψ_r, ψ_r', $\frac{d\psi}{d\mu}$, and $\frac{d\psi'}{d\mu}$ define boundary data on Γ for functions ψ that admit representation (31). The set of all functions in the space $W_2^2(R^1 \setminus \Gamma)$ satisfying boundary conditions will be denoted by $W_2^2(R^1 \setminus \Gamma; d\mu)$. For functions $\psi, \varphi \in W_2^2(R^1 \setminus \Gamma; d\mu)$, it was proved in [8] that Green's first and second formulas hold with boundary values of ψ and φ on Γ.

Green's first formula is

$$
(-\psi'', \varphi)_{L_2(R^1)} = (\psi', \varphi')_{L_2((R^1)} + \int_\Gamma \left[\frac{d\psi'}{d\mu}\overline{\varphi_r} + \psi_r'\frac{d\overline{\varphi}}{d\mu} \right] d\mu.
\tag{32}
$$

Green's second formula is

$$
(-\psi'', \varphi)_{L_2(R^1)} - (\psi, -\varphi'')_{L_2((R^1)} = \int_\Gamma \left[\frac{d\psi'}{d\mu}\overline{\varphi_r} + \psi_r'\frac{d\overline{\varphi}}{d\mu} - \psi_r\frac{d\overline{\varphi'}}{d\mu} - \frac{d\psi}{d\mu}\overline{\varphi_r'} \right] d\mu
$$

$$
= \omega(\Gamma\psi, \Gamma\varphi) = <\hat{\Gamma}_1\psi, \hat{\Gamma}_2\varphi> - <\hat{\Gamma}_2\psi, \hat{\Gamma}_1\varphi>,
$$

$$
\hat{\Gamma}_1\psi = \mathrm{col}\,(\frac{d\psi'}{d\mu}, \frac{d\psi}{d\mu}), \qquad \hat{\Gamma}_2 = \mathrm{col}\,(\psi_r, -\psi_r'). \tag{33}
$$

Green's second formula allows to consider different self-adjoint boundary conditions that are similar to one-point conditions considered in Section 2. They include the following boundary conditions that correspond to δ'-interaction on Γ:

$$
\frac{d\psi'(x)}{d\mu} = 0, \quad \frac{d\psi(x)}{d\mu} = \beta(x)\psi_r'(x), \ x \in \Gamma.
\tag{34}
$$

Here, the real-valued function β is defined on Γ and is absolutely integrable with respect to measure μ. The function β defines the intensity of the δ'-interaction on Γ.

7 Spectral properties of Schrödinger operator with δ'-interaction

The boundary conditions (34) define a Schrödinger operator with δ'-interaction on Γ. The definition domain of such an operator $L_{\Gamma,\beta}$ consists of all functions in the space $W_2^2(R^1 \setminus \Gamma; d\mu)$ that satisfy the boundary conditions (34). The operator acts on such a function ψ by $L_{\Gamma,\beta}\psi = -\psi''(x)$, $x \notin \Gamma$. This operator is Hermitian in virtue of Green's second formula (33). It was proved in [8] that it is self-adjoint. This is the following result.

Theorem 2. *Let Γ be a bounded closed subset of the real line, having Lebesgue measure zero. Let a real-valued function β be absolutely integrable on Γ with respect to a Radon measure μ. The Schrödinger operator $L_{\Gamma,\beta}$ is self-adjoint on the space $L_2(R^1)$ and defines a δ'-interaction on Γ. The negative spectrum of the operator $L_{\Gamma,\beta}$ is discrete.*

Proof. Since we work here with the abstract Definition 3 of a Schrödinger operator with δ'-interaction on Γ, the proof from [8] needs to be modified in view of this definition. The Schrödinger operator $L_{\Gamma,\beta}$ is self-adjoint. This is proved in [8] for Γ being a Cantor set and a Hausdorff measure on Γ. This proof is correct for the general case of δ'–interaction on a set Γ with a measure μ. Let us consider an operator $L_{\Gamma,\beta}^{(a,b)}$ in a space $L_2(a,b)$, where the interval (a,b) contains the set Γ. The domain of operator $L_{\Gamma,\beta}^{(a,b)}$ consists of the restrictions on the interval (a,b) of all functions of $W_2^2(R^1 \setminus \Gamma, d\mu)$, that satisfy boundary conditions (34) and also boundary conditions $\psi(a) = 0$, $\psi'(b) = 0$ at the endpoints of interval. The action of the operator $L_{\Gamma,\beta}^{(a,b)}$ on these functions ψ leads to $-\psi''(x)$ with $x \notin \Gamma$. Let us show that the operator $L_{\Gamma,\beta}^{(a,b)}$ is self–adjoint in the space $L_2(a,b)$. For this, at the beginning, let us show that the range of values of operator $L_{\Gamma,\beta}^{(a,b)}$ is the whole space $L_2(a,b)$. In fact, since $\frac{d\psi'(x)}{d\mu} = 0$, then because of (31) and the boundary condition $\psi'(b) = 0$: $\psi'(x) = \int\limits_x^b h(s)\,ds$, where $h(x) = L_{\Gamma,\beta}^{(a,b)}\psi(x) = -\psi''(x)$. Therefore, for any $h \in L_2(a,b)$ we have $\psi'(x) = \psi_r'(x)$, and considering (31), boundary conditions (34) and condition $\psi(a) = 0$ with $x \notin \Gamma$, we have

$$\psi(x) = \int\limits_a^x \psi'(s)\,ds + \int\limits_a^x \beta(s)\psi'(s)\,d\mu(s) = \int\limits_a^b \mathcal{G}(x,s)h(s)\,ds \tag{35}$$

where $\mathcal{G}(x,s) = \min(x,s) - a + \int\limits_a^{\min(x,s)} \beta(\xi)d\mu(\xi)$. The representation (35) shows that the operator $[L_{\Gamma,\beta}^{(a,b)}]^{-1}$ is an integral bounded Hermitian operator in the space $L_2(a,b)$. Therefore, the operator $L_{\Gamma,\beta}^{(a,b)}$ is self–adjoint in the space $L_2(a,b)$.

Let us consider the direct sum of self–adjoint operators L_D, $L_{\Gamma,\beta}^{(a,b)}$, L_N : $L = L_D \oplus L_{\Gamma,\beta}^{(a,b)} \oplus L_N$ in the space $L_2(R^1) = L_2(-\infty, a) \oplus L_2(a,b) \oplus L_2(b, \infty)$. Here, the self–adjoint operator L_D is defined in the space $L_2(-\infty, a)$ by the differential expression $-\frac{d^2}{dx^2}$ on functions of the space $W_2^2(-\infty, a)$, that satisfy Dirichlet boundary condition $\psi(a) = 0$. The self–adjoint operator L_N is defined in the space $L_2(b, \infty)$ by differential expression $-\frac{d^2}{dx^2}$ and Neumann boundary condition $\psi'(b) = 0$. The self–adjoint operator $L_{\Gamma,\beta}^{(a,b)}$ is defined above in the space $L_2(a,b)$. It is easy to see, that the symmetric operator $L_{\Gamma,\beta}$ is a finite rank perturbation of the self–adjoint operator L in the space $L_2(R^1)$ and corresponds to self–adjoint boundary conditions $\psi(a-0) = \psi(a+0)$,

$\psi'(a-0) = \psi'(a+0), \psi(b-0) = \psi(b+0), \psi'(b-0) = \psi'(b+0)$. Therefore, [20] the operator $L_{\Gamma,\beta}$ is self–adjoint in the space $L_2(R^1)$. Since the operator $[L_{\Gamma,\beta}^{(a,b)}]^{-1}$ is compact and the operators L_D and L_N have absolutely continuous spectrum $[0, +\infty)$ and the spectrum of the operator $L_{\Gamma,\beta}^{(a,b)}$ is discrete with only possible limit point $\lambda = \infty$ then the spectrum of the operator $L = L_D \oplus L_{\Gamma,\beta}^{(a,b)} \oplus L_N$ and consequently the spectrum of the operator $L_{\Gamma,\beta}$ can be only discrete on the negative half-axis since the self–adjoint operator $L_{\Gamma,\beta}$ is a finite rank perturbation of the operator L.

On the other hand, the domain $D(L_{\Gamma,\beta})$ possesses the properties required in Definition 3. Indeed, it follows from representation (31) that if a function $\psi(x)$ has boundary values on Γ, then the same is true for the function $\chi(x) \cdot \psi(x)$. The boundary data for the function $\chi(x) \cdot \psi(x)$ coincide with the boundary data for the function $\psi(x)$ multiplied by the function χ, that is, $\frac{d(\chi\psi)}{d\mu} = \chi\frac{d\psi}{d\mu}$, which means that $(\chi\psi)_r = \chi\psi_r$, etc. This shows that, if $\psi \in D(L_{\Gamma,\beta})$, then $\chi\psi \in D(L_{\Gamma,\beta})$.

Hence, the self-adjoint operator $L_{\Gamma,\beta}$ describes a local interaction on Γ according to Definition 1. Since the function χ has trivial boundary data and $\frac{d\chi}{d\mu} = 0$, $\frac{d\chi'}{d\mu} = 0$, $\chi'_r = 0$, $\chi_r = \chi$, it follows that this function satisfies boundary conditions (31) and, consequently, $\chi \in D(L_{\Gamma,\beta})$. It now follows from Definition 3 that the self-adjoint operator $L_{\Gamma,\beta}$ defines a δ'-interaction on the set Γ. $\square$

In order to extend the results of Theorem 1 to a general case, we will need the following definition.

Definition 5. We say that a real-valued function β defined on a set Γ with a measure μ assumes negative values on an infinite number of subsets of Γ if for any natural N there exists $\varepsilon > 0$ and a collection of closed measurable nonintersecting subsets $\Gamma_k \subset \Gamma$, $\mu(\Gamma_k) > 0$, $k = 1, ..., N$, such that the function $\beta(x)$ assumes strictly negative values on Γ_k, $\beta(x) \leq -\varepsilon$, $x \in \Gamma_k$, $k = 1, ..., N$.

Theorem 3. *Let a real-valued function β, defined on a closed bounded set Γ of Lebesgue measure zero, be absolutely integrable with respect to a Radon measure μ and assume negative values on an infinite number of subsets of Γ. Then the Schrödinger operator $L_{\Gamma,\beta}$ with δ'-interaction on Γ, having intensity β, is a self-adjoint operator on the space $L_2(R^1)$ and has an infinite number of negative eigenvalues, $\lambda_n \to -\infty$.*

Proof. The proof is similar to the proof of Theorem 1. Let the conditions of the theorem be satisfied. Then, by Theorem 2, the operator $L_{\Gamma,\beta}$ is self-adjoint on $L_2(R^1)$ and the negative spectrum of the operator $L_{\Gamma,\beta}$ is discrete. Let us show that the operator $L_{\Gamma,\beta}$ has an infinite number of negative eigenvalues. To this end, it is sufficient to show that there exists an N–dimensional subspace $\mathcal{L}_N$ of the domain of $L_{\gamma,\beta}$ such that $(L_{\Gamma,\beta}u, u) < 0$, for any $u \in \mathcal{L}_N$, $u \neq 0$ for any natural N. Fix N. By the conditions of the theorem there exist N non-intersecting closed subsets $\Gamma_k \subset \Gamma$, $\mu(\Gamma_k) > 0$, and $\varepsilon > 0$ such that $\beta(x) \leq -\varepsilon$ for $x \in \Gamma_k$, $k = 1, ..., N$. Consider analogues of the test functions of Section 4. Since the number of subsets Γ_k is finite, they are closed and nonintersecting, there is $\delta > 0$ such that all δ-neighborhoods $\mathcal{U}_\delta(\Gamma_k) = \{y : |y-x| < \delta, x \in \Gamma_k\}$ of the sets Γ_k are also pairwise nonintersecting. Let us construct a test function for each set Γ_k as follows. Consider the function $\chi_k(x) \in C_0^\infty(R^1)$ that equals 1 on Γ_k, takes values between 0 and 1, and equals to zero outside of $\mathcal{U}_\delta(\Gamma_k)$. Such a step function can be constructed as usual by making a smooth function from the characteristic function of the set $\mathcal{U}_{\frac{\delta}{2}}(\Gamma_k)$. As a candidate for the test function, we take

$$\hat{t}_k(x; \beta, \Gamma_k, \delta) = \int_a^x \chi_k(\xi)\,d\xi + \int_{(a,x)} \beta(\xi)\chi_k(\xi)\,d\mu(\xi), \tag{36}$$

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where the number a is chosen so that all bounded sets $\mathcal{U}_\delta(\Gamma_k)$, $k = 1, ..., N$, would lie to the right of the point a. For x that lie on the right of the set Γ, this function takes the constant value c_k. While the function t_k does not belong to the space $L_2(R^1)$, we can turn it into a function with compact support using two parabolas on the interval $[l, l + 2r]$ that lies to the right of Γ. We thus get the test function

$$t_k(x; \beta, \Gamma_k, \delta, l, r) = \begin{cases} \hat{t}, & x \leq l, \\ -\frac{c_k}{2r^2}(l - x)^2 + c_k, & l \leq x \leq l + r, \\ \frac{c_k}{2r^2}(l + 2r - x)^2, & l + r \leq x \leq l + 2r, \\ 0, & l + 2r < x. \end{cases} \tag{37}$$

Here, the parameters l and r may depend on k.

Proposition 4. *The main properties of the test functions t_k (37) are the following:*

1^0 $t_k \in \mathcal{D}(L_{\Gamma,\beta})$ *if* $\Gamma \subset (-\infty, l)$.

2^0 *By choosing δ sufficiently small and r sufficiently large, we have*

$$(L_{\Gamma,\beta} t_k, t_k) \leq -\frac{1}{8}\varepsilon \mu(\Gamma_k), \tag{38}$$

that is, the quadratic form takes negative values.

3^0 *The quadratic form of the linear combination $t = \sum_{k=1}^{N} a_k \cdot t_k$ of test functions that satisfy the condition 1^0, if l_k and r_k are chosen so that the intervals $[l_k, l_k + 2r_k]$ are pairwise disjoint, takes negative values,*

$$(L_{\Gamma,\beta} t, t) = \sum_{k=1}^{N} |a_k|^2 (L_{\Gamma,\beta} t_k, t_k) \leq -\frac{1}{8}\varepsilon \min_k \mu(\Gamma_k) \sum_{k=1}^{N} |a_k|^2 < 0. \tag{39}$$

If these three conditions are satisfied, then the proof is finished by applying the variational minimax principle [26] as in the proof of Theorem 1.

Let us now prove that test functions satisfy properties 1^0—3^0. The first property is clearly satisfied by the construction of t_k and $\hat{t}_k$ in (36) and (37) and the definition of the operator $L_{\Gamma,\beta}$. The second property is most important. Since the function β is absolutely integrable on Γ with respect to the Radon measure μ and $0 \leq \chi_k \leq 1$, we see that there exists small δ such that

$$\left| \int_{\mathcal{U}_\delta(\Gamma_k) \cap \Gamma} \beta(\xi)\chi_k(\xi) \, d\mu(\xi) - \int_{\Gamma_k} \beta(\xi) \, d\mu(\xi) \right| < \frac{1}{2}\varepsilon \mu(\Gamma_k). \tag{40}$$

Moreover, since the set Γ has Lebesgue measure zero, there exists a small δ such that the following estimate holds for the Lebesgue measure of the set $\mathcal{U}_\delta(\Gamma_k)$:

$$|\mathcal{U}_\delta(\Gamma_k)| \leq \frac{1}{4}\varepsilon \mu(\Gamma_k). \tag{41}$$

If inequalities (40) and (41) hold, then the constant c_k, which is equal to the value of the function $\hat{t}$ for large k, satisfies the estimate

$$|c_k| \leq (\frac{3}{4}\varepsilon + ||\beta||_{L_1(\Gamma, d\mu)})\mu(\Gamma_k). \tag{42}$$

By choosing r_k large enough, we have

$$\int\limits_{l_k}^{l_k+2r_k} |t'(x)|^2\, dx \le \frac{1}{8}\varepsilon\mu(\Gamma_k). \tag{43}$$

In virtue of Green's first formula (32), since the function t_k satisfies the boundary conditions (34) and because $t'_k(x) = \chi_k(x)$ for $x \le l_k$, we have

$$(L_{\Gamma,\beta}t_k, t_k) = \int\limits_{a}^{l_k} |\chi_k(x)|^2\, dx + \int\limits_{l_k}^{l_k+2r_k} |t'_k(x)|^2\, dx + \int\limits_{\Gamma} \beta(\xi)|\chi_k(\xi)|^2\, d\mu(\xi). \tag{44}$$

The first integral $\mathcal{I}_1$ in (44) can be estimated in terms of the Lebesgue measure $\mathcal{U}_\delta(\Gamma_k)$, since values of the function $\chi_k(x)$ belong to the interval $[0,1]$. The second integral $\mathcal{I}_2 = \frac{2}{3}c_k^2 \cdot r_k^{-1}$ can be explicitly calculated, since the function $t'_k(x)$ on the interval $[l_k, l_k + 2r_k]$ consists of two parabolas by (37). The third integral $\mathcal{I}_3$ in (44) can be estimated as follows:

$$\mathcal{I}_3 = \int\limits_{\Gamma_k} \beta(\xi)\, d\xi + \left| \int\limits_{\mathcal{U}_\delta(\Gamma_k)} \beta(\xi)\chi_k^2(\xi)\, d\mu(\xi) - \int\limits_{\Gamma_k} \beta(\xi)\, d\mu(\xi) \right|.$$

Since, by choosing sufficiently small δ and sufficiently large r_k we can satisfy estimates (40)–(43), we see that the quadratic form $(L_{\Gamma,\beta}t_k, t_k)$ is negative, i.e., inequality (38) is satisfied.

Consider now property 3^0. Since the intervals $(l_k, l_k + 2r_k)$ and the regions $\mathcal{U}_\delta(\Gamma_k)$ are mutually disjoint, we have that $(L_{\Gamma,\beta}t_k, t_j) = 0$ for $k \ne j$. This leads to property (39). $\qquad\square$

For nonlocal interactions we may get a behaviour different from the one in the local case. We illustrate this fact by the following example.

Example 1. It is not possible that the same function is eigenfunction with negative eigenvalue of two different Schrödinger operators with local δ and δ' interactions. This is not true for nonlocal point interactions.

Indeed, let A_1 be the self–adjoint operator in $L_2(R^1)$ that corresponds to the two–point nonlocal interaction in the points $x_1 = -1$ and $x_2 = 1$ described by following self–adjoint boundary conditions

$$\psi'(x_j + 0) - \psi'(x_j - 0) = 0,$$
$$\psi'(x_j + 0) + \psi'(x_j - 0) + \psi'(x_1 + 0) - \psi'(x_1 - 0) + \psi(x_2 + 0) - \psi(x_2 - 0) = 0, \; j = 1, 2. \tag{45}$$

There exists a unique negative eigenvalue $-\lambda_0^2$ of the operator A_1 where the number λ_0 is the positive root of the characteristic equation $\lambda_0 = 1 + \tanh \lambda_0$, $\lambda_0 \approx 1.968$. The eigenfunction $\psi_0(x)$ is odd $\psi_0(-x) = -\psi_0(x)$ and has the form:

$$\psi_0(x) = \begin{cases} -\frac{\sinh \lambda_0 x}{\cosh \lambda_0}, & 0 \le x < 1, \\ e^{-\lambda_0(x-1)}, & 1 < x < +\infty \end{cases} \tag{46}$$

Let us consider the self–adjoint operator A_2 that corresponds to the local δ' interaction in the points $x_1 = -1$ and $x_2 = 1$ with intensity $\beta = -1$. The domain of the operator A_2 is given by self–adjoint conditions

$$\psi'(x_j + 0) - \psi'(x_j - 0) = 0,$$
$$\psi(x_j + 0) - \psi(x_j - 0) = -\psi'(x_j), \; j = 1, 2. \tag{47}$$

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It is easy to check that the function $\psi_0(x)$ of (46) satisfies the boundary conditions (47), i.e. is an eigenfunction of the operator A_2. However, there exists one more even eigenfunction $\psi_1(x) = \psi_1(-x)$ of the form

$$\psi_1(x) = \begin{cases} -\frac{\cosh \lambda_1 x}{\sinh \lambda_1}, & 0 < x < 1, \\ e^{-\lambda_1(x-1)}, & 1 < x < +\infty \end{cases} \tag{48}$$

with negative eigenvalue $-\lambda_1^2$ where λ_1 is the positive root of the equation $\lambda_1 = 1 + \coth \lambda_1$, $\lambda_1 \approx 2.03$.

8   Deficiency subspaces

In this section we give for arbitrary closed subsets Γ of R^1 with Lebesgue measure zero the deficiency subspaces of the operator $L_{min,\Gamma}$. This result can be used for the construction of Hamiltonians describing an interaction which takes place inside Γ.

First we fix some notation and consider any symmetric operator S in any complex Hilbert space $\mathcal{H}$ such that the deficiency subspaces $\mathrm{ran}(S \pm i)^\perp$ of S have the same Hilbert space dimension. For every unitary transformation $U : \mathrm{ran}(S + i)^\perp \longrightarrow \mathrm{ran}(S - i)^\perp$ put

$$\begin{aligned} D(S_U) &:= \{f + Uf + h : f \in \mathrm{ran}(S + i)^\perp, h \in D(\bar{S})\}, \\ S_U &:= S^* \lceil D(S_U). \end{aligned} \tag{49}$$

By von Neumann's first and second formula, the mapping $U \mapsto S_U$ from the set of unitary transformations $U : \mathrm{ran}(S + i)^\perp \longrightarrow \mathrm{ran}(S - i)^\perp$ onto the set of self–adjoint extensions of S is bijective. Moreover every $f \in D(S^*)$ can be uniquely represented as

$$f = f_+ + f_- + h, \quad f_\pm \in \mathrm{ran}(S \pm i)^\perp, h \in D(\bar{S}).$$

Thus

$$f_- = Uf_+, \text{ if } f_\pm \in \mathrm{ran}(S \pm i)^\perp, h \in D(\bar{S}) \text{ and } f_+ + f_- + h \in D(S_U). \tag{50}$$

Let D be a closed linear subspace of $\mathrm{ran}(S + i)^\perp$. Put

$$\begin{aligned} D(S_U^D) &:= \{f + Uf + h : f \in \mathrm{ran}(S + i)^\perp \cap D^\perp, h \in D(\bar{S})\}, \\ S_U^D &:= S^* \lceil D(S_U^D). \end{aligned} \tag{51}$$

$f \in \mathrm{ran}(S_U^D + i)^\perp$ if and only if $f \in \mathrm{ran}(S + i)^\perp$ and

$$f \perp (S^* + i)(f_+ + Uf_+) = 2if_+, \quad f_+ \in \mathrm{ran}(S + i)^\perp \cap D^\perp.$$

Thus $\mathrm{ran}(S_U^D + i)^\perp = D$ and we have proved the following lemma:

Lemma 5. *Let $V : \mathrm{ran}(S + i)^\perp \longrightarrow \mathrm{ran}(S - i)^\perp$ be any linear mapping such that $Vf = Uf$ for all $f \in \mathrm{ran}(S + i)^\perp \cap D^\perp$ and $V\lceil D$ is a unitary mapping from D onto $\{Uf : f \in D\}$. Then V is a unitary mapping from $\mathrm{ran}(S + i)^\perp$ onto $\mathrm{ran}(S - i)^\perp$, $S_U^D = S_V^D$ is a restriction of S_U and S_V and*

$$\mathrm{ran}(S_V^D + i)^\perp = D \tag{52}$$

By Krein's formula and (52), $(S_U + i)^{-1} - (S_V + i)^{-1}$ is a finite rank operator with rank $\dim D$, provided D is finite dimensional. By Weyl's essential spectrum theorem, the Birman-Kuroda theorem, and a theorem by Krein this implies the following result:

Lemma 6. *Let S be a symmetric operator in the Hilbert space $\mathcal{H}$, D a finite dimensional subspace of $\mathrm{ran}(S+i)^{\perp}$ and U and V unitary transformations from $\mathrm{ran}(S+i)^{\perp}$ onto $\mathrm{ran}(S-i)^{\perp}$ which coincide on $D^{\perp} \cap \mathrm{ran}(S+i)^{\perp}$. Then the self–adjoint extensions S_U and S_V (cf. (49)) have the same essential and the same absolutely continuous spectrum. and the number, counting multiplicities, of eigenvalues of S_V below the minimum of the essential spectrum of S_V is less than or equal to $\dim D$.*

Now let us consider explicite examples. Let Γ be a closed subset of R^d with Lebesgue measure zero and $2\alpha \in \mathbb{N}$. Let S be the symmetric operator in $L^2(R^d)$ defined as follows:

$$
\begin{aligned}
D(S) &:= C_0^{\infty}(R^d \setminus \Gamma), \\
Sf &:= (-\Delta)^{\alpha} f, \quad f \in D(S).
\end{aligned}
$$

Based on ideas in [26] and with the aid of the theorem on the spectral synthesis in Sobolev spaces [25] one has determined the deficiency subspaces

$$
\mathrm{ran}(S-z)^{\perp}, \quad z \in \mathbb{C} \setminus [0, \infty),
$$

of the operator S, cf. [10], Example 2.8. In order to formulate this result in the case we are interested in, i.e. $d = 1 = \alpha$, we use the following notation:

$$
g_z(x) := \frac{i}{2\sqrt{z}} e^{i\sqrt{z}|x|}, \quad x \in R^1, z \in \mathbb{C} \setminus [0, \infty), \tag{53}
$$

where the square root has to be chosen such that the imaginary part of $\sqrt{z}$ is positive, $\mathcal{M}_{\Gamma}$ denotes the set of positive Radon measures on R^1 with compact support in Γ and

$$
\mathcal{T}_{z,\Gamma} := \{g_z * \mu : \mu \in \mathcal{M}_{\Gamma}\} \cup \{(g_z * \nu)' : \nu \in \mathcal{M}_{\Gamma}\}. \tag{54}
$$

Since every finite positive Radon measure μ on R^1 belongs to the Sobolev space $W_2^{-1}(R^1)$, we get the following result:

Lemma 7. (cf. [10], Example 2.8) *Let Γ be a closed subset of R^1 with Lebesgue measure zero. Then $\mathcal{T}_{z,\Gamma}$, defined by (54), is a total subset of the deficiency subspaces $\mathfrak{N}_{z,\Gamma} \equiv \mathrm{ran}(L_{\min,\Gamma} - \overline{z})^{\perp} \equiv \ker(L_{\max,\Gamma} - z)$, i.e. $\mathfrak{N}_{z,\Gamma}$ is the closure of the span of $\mathcal{T}_{z,\Gamma}$.*

Let (μ_n) be a sequence of finite positive Radon measures converging weakly to the finite positive Radon measure μ. Then, by the dominated convergence theorem, the sequences of the Fourier transforms of $(g_z * \mu_n)$ and $((g_z * \mu_n)')$ converge in $L^2(R^1)$ to the Fourier transform of $g_z * \mu$ and $(g_z * \mu)'$, respectively. Hence the sequences $(g_z * \mu_n)$ and $((g_z * \mu_n)')$ converge in $L^2(R^1)$ to $g_z * \mu$ and $(g_z * \mu)'$, respectively. Moreover for every finite positive Radon measure on R^1 there exist positive Radon measures μ_n, $n \in \mathbb{N}$, such that the support of μ_n is a finite subset of the support of μ for every $n \in \mathbb{N}$ and the sequence (μ_n) converges weakly to μ. By Lemma 7, this implies that

$$
\{g_z(x - \gamma) : \gamma \in \Sigma\} \cup \{g_z'(x - \gamma) : \gamma \in \Sigma\}
$$

is a total subset of the deficiency subspace $\mathfrak{N}_{z,\Gamma}$, if Σ is dense in Γ. Moreover if γ is not an isolated point of Γ, then $g_z(x - \gamma)$ and $g_z'(x - \gamma)$ belong to the closure of the span of the set $\{g_z(x - \gamma) : \gamma \in \Sigma\}$. Thus we get the following result:

Proposition 5. *Let Γ be a closed subset of R^1 with Lebesgue measure zero. Then for every $z \in \mathbb{C} \setminus [0, \infty)$*

$$B(\Sigma_1, \Sigma_2) = \{g_z(x - \gamma) : \gamma \in \Sigma_1\} \cup \{g_z'(x - \gamma) : \gamma \in \Sigma_2\} \tag{55}$$

is a total subset of the deficiency subspace $\mathfrak{N}_{z,\Gamma}$, if, and only if, Σ_1 is dense in Γ and Σ_2 contains the set of all isolated points of Γ.

Proof. Let us show that the conditions on Σ_1 and Σ_2 are necessary for totality of the set $B(\Sigma_1, \Sigma_2)$ in $\mathfrak{N}_{z,\Gamma}$. Let Σ_1 be not dense in Γ. Then, there exists a partition $\Gamma = \Gamma_1 \cup \Gamma_2$ on two not empty, not intersecting closed subsets Γ_1 and Γ_2 and $\Sigma_1 \subset \Gamma_1$. In this case,

$$\mathfrak{N}_{z,\Gamma} = \mathfrak{N}_{z,\Gamma_1} \dot{+} \mathfrak{N}_{z,\Gamma_2},$$

where the sum is direct and corresponding to this sum the skew projectors $P_j : \mathfrak{N}_{z,\Gamma} \to \mathfrak{N}_{z,\Gamma_j}$, $j = 1, 2$ are bounded operators. Let us now show that in this case the set $B(\Gamma_1, \Gamma)$ which is larger than $B(\Sigma_1, \Sigma_2)$ will not be total in the deficiency space $\mathfrak{N}_{z,\Gamma}$. If $B(\Gamma_1, \Gamma)$ would be total in $\mathfrak{N}_{z,\Gamma}$ then the set $B(\Gamma_2) = \{g_z'(x - \gamma) : \gamma \in \Gamma_2\}$ would be total in $\mathfrak{N}_{z,\Gamma_2}$. However, it is not possible. In fact, a linear continuous with respect to the metric of $L_2(R^1)$ functional $e(f) = \int\limits_{R^1} f(x)\, dx$ is defined on the whole $\mathfrak{N}_{z,\Gamma_2}$ and is equal to zero on $\mathrm{span}B(\Gamma_2)$ but it is equal to $-\dfrac{1}{z}$ on a function $g_z(x - \gamma) \in \mathfrak{N}_{z,\Gamma_2}$. One can prove that if Σ_2 does not contain all isolated points x_0 in Γ then even $B(\Gamma, \Gamma \setminus \{x_0\})$ can not be total in $\mathfrak{N}_{z,\Gamma}$. In this case, the deficiency space $\mathfrak{N}_{z,\{x_0\}}$ is two-dimensional and a skew projection $\mathrm{span}B(\Gamma, \Gamma \setminus \{x_0\})$ on $\mathfrak{N}_{z,\{x_0\}}$ is a one-dimensional subspace. $\square$

Example 2. Let Γ be a closed subset of R^1 with Lebesgue measure zero and put $S := L_{min,\Gamma}$. As pointed out in lemma 6 one may get far reaching results on the spectral properties of one self–adjoint extension S_V of S with the aid of another self–adjoint extension S_U of S. Since one knows the spectral properties of the free quantum mechanical Hamiltonian, it is interesting to determine the unitary mapping U such that S_U is the free quantum mechanical Hamiltonian, i.e.

$$S_U \psi(x) = -\psi''(x), \quad \psi \in D(S_U) = W_2^2(R^1). \tag{56}$$

Passing to Fourier transforms one sees that $g_{-i} * \mu - g_i * \mu \in W_2^2(R^1)$ and $(g_{-i} * \mu)' - (g_{+i} * \mu)' \in W_2^2(R^1)$ for every $\mu \in \mathcal{M}_\Gamma$. By (50), this implies that

$$U g_{-i} * \mu = -g_i * \mu \text{ and } U((g_{-i} * \mu)') = -(g_i * \mu)', \quad \mu \in \mathcal{M}_\Gamma. \tag{57}$$

Now fix $\mu \in \mathcal{M}_\Gamma$ and $\alpha \in \mathbb{S}^1$. By Lemma 5 there exists a unique self–adjoint extension A of S such that

$$\psi_0 := (g_{-i} * \mu)' + \alpha (g_i * \mu)' \in D(A) \tag{58}$$

and A and S_U have a common restriction T such that $\mathrm{ran}(T + i)^\perp$ is spanned by $(g_{-i} * \mu)'$. By Lemma 6, A and S_U have the same essential spectrum and the same absolutely continuous spectrum and hence

$$\sigma_{ess}(A) = [0, \infty) = \sigma_{ac}(A), \tag{59}$$

and the number, counting multiplicities, of negative eigenvalues of A is less than or equal to one.

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